

# Two-Neutrino Double Beta Decay: Critical Analysis

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## Abstract

We have performed a critical analysis of different approximation schemes for the calculation of two-neutrino double beta decay ( $2\beta 2\nu$ -decay) matrix elements. For that purpose the time integral representation of the  $2\beta 2\nu$ -decay matrix element has been used. We have shown that within the single-particle approximation of the nuclear Hamiltonian the  $2\beta 2\nu$ -decay matrix element is equal to zero because of the mutual cancelation of the direct and cross terms. The quasiboson approximation (QBA) and renormalized QBA (RQBA) schemes imply for the  $2\beta 2\nu$ -decay transition operator to be a constant, if one requires the equivalence of initial and final Quasiparticle Random Phase Approximation (QRPA) and renormalized QRPA (RQRPA) Hamiltonians. It means that  $2\beta 2\nu$ -decay is a higher order process in the boson expansion of the nuclear Hamiltonian and its higher order boson approximations are important. The equivalence of the initial and final QRPA and RQRPA

Hamiltonians has been discussed within the QBA and RQBA, respectively. We have found that the mismatching of both Hamiltonians is getting worse with increasing strength of particle-particle interaction especially in the case of QRPA Hamiltonians. It is supposed to be one of the reasons of the extreme sensitivity of the  $2\beta 2\nu$ -decay matrix element to the residual interaction appearing in explicit calculations involving the intermediate nucleus. Further, the Operator Expansion Method (OEM) has been reconsidered and new  $2\beta 2\nu$ -decay transition operators have been rederived in a consistent way. The validity of the OEM approximation has been discussed in respect to the other approximation schemes. The OEM combined with QRPA or RQRPA ground state wave functions reflects sensitively the instabilities incorporated in the considered ground states. Therefore, the predicting power of the OEM should be studied with help of the other ground state wave functions, e.g. shell model ones.

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## I. INTRODUCTION

The two-neutrino double beta decay ( $2\beta 2\nu$ -decay) is a second order process of the weak interaction, which is allowed by the Standard model [1]. In  $2\beta 2\nu$ -decay the nucleus  $(A, Z)$  undergoes the transition to nucleus  $(A, Z+2)$  with emission of two electrons and two antineutrinos. This rare process is already well established experimentally for a couple of isotopes. The inverse half-life of  $2\beta 2\nu$ -decay is free of unknown parameters on the particle physics side and is expressed as a product of a phase-space factor and the relevant  $2\beta 2\nu$ -decay nuclear matrix element. Since the phase-space factor can be calculated with the desired accuracy, the experimental studies of  $2\beta 2\nu$ -decay give us directly the value of the  $2\beta 2\nu$ -decay nuclear matrix element. In this way  $2\beta 2\nu$ -decay offers a sensitive test of nuclear structure calculations. The calculation of the  $2\nu\beta\beta$ -decay nuclear transition continues to be challenging in view of the smallness of the predicted nuclear matrix elements and the fact that the mechanism which is leading to the suppression of these matrix elements is still not completely understood.

The proton-neutron Quasiparticle Random Phase Approximation (pn-QRPA) has been the most frequently used nuclear structure method for evaluating  $2\beta 2\nu$ -decay rates because of the remarkable success achieved in revealing the suppression mechanism of  $2\beta 2\nu$ -decay matrix elements [2]–[4]. However, the extreme sensitivity of the  $2\beta 2\nu$ -decay matrix elements on the pn  $1^+$  particle-particle matrix element as well as the collapse of the QRPA solution in the physically acceptable region of the particle-particle strength of the nuclear Hamiltonian renders it difficult to make definite rate predictions.

Some attempts have been done to overcome the above drawbacks, e.g. by including higher order RPA corrections [5], particle number projection [6,7] and proton-neutron pairing [8] in the theory. However, none of these modifications of the QRPA prevents the collapse and inhibit the nuclear matrix element to cross zero close to the physical value of the particle-particle force. Recently, Toivanen and Suhonen have proposed a proton-neutron renormalized QRPA (pn-RQRPA) [9], which goes beyond the QRPA and takes into account

the Pauli exclusion principle in an approximate way. It has been shown that the above phenomena could be connected with the limitation of the QRPA approach - the quasiboson approximation, which violates the Pauli exclusion principle. The renormalized quasiboson approximation (QBA), on which the pn-RQRPA is based, inhibits the collapse of the pn-RQRPA solution for a physical value of the particle-particle interaction strength. In addition, the calculated  $2\beta 2\nu$ -decay nuclear matrix elements via pn-RQRPA have been found significantly less sensitive to the particle-particle force within its physical values in respect to those obtained by the pn-QRPA [9]– [12]. This behavior has been confirmed also within the renormalized QRPA with proton-neutron pairing (full-RQRPA) [10].

In spite of the advantages, of the renormalized QRPA over the QRPA, the RQRPA can not be considered as the ultimate solution for the calculation of the  $2\beta 2\nu$ -decay process. Several shortcomings still plague the RQRPA: i) The Ikeda sum rule is violated [12,13]. ii) There are two sets of the intermediate nuclear states in the calculation generated respectively from initial and final nuclei, which do not coincide with each other. iii) The Pauli exclusion principle is taken into account only in an approximate way. All these leaps of faith of the RQRPA approach have common origin. It is the particle number non-conservation. The effect of these shortcomings on the  $2\beta 2\nu$ -decay amplitude is well understood.

A longstanding problem of large discrepancies between the values of the predicted and calculated  $2\beta 2\nu$ -decay matrix elements has led to a development of alternative methods, e.g. one of them is the Operator Expansion Method (OEM) [14]– [23]. The OEM tries to avoid the necessity of evaluating the sum over the intermediate nuclear states. The price paid for it is that one has now to deal with more than two-body operators and commutators involving kinetic energy terms in the commutator expansion of the  $2\beta 2\nu$ -decay transition operator. Recently, this method has been reconsidered [23] and it has been shown that the previous derivation [14]– [21] of the OEM-potential was not consistent. The OEM-calculation with a consistent OEM-potential combined with the pn-RQRPA ground state wave functions (OEM+RQRPA) has exhibited a large sensitivity on the strength of the particle-particle force within its physical values [23]. There is a speculation that the approximations of the

pn-RQRPA method are responsible for this behavior. Therefore, detail OEM+RQRPA calculations are expected to be helpful for solving the problem of stability of the  $2\beta 2\nu$ -decay matrix element in respect to model parameters.

Till now no consistent many body approach is available for the calculation of the many-body Green function governing the  $2\beta 2\nu$ -decay process, because of the computational complexity of the problem. Therefore we can not avoid the introduction of different approximation schemes in the evaluation of the nuclear matrix elements. Nevertheless, we can try to understand the limitations of the different approximation schemes. The aim of this work is to perform a critical analysis of the QBA and renormalized QBA schemes by using the time integral representation of the  $2\beta 2\nu$ -decay nuclear matrix element and to discuss the validity of the QRPA, RQRPA and OEM+RQRPA calculations.

## II. $2\beta 2\nu$ -DECAY NUCLEAR MATRIX ELEMENT.

If the two-nucleon mechanism for the  $2\beta 2\nu$ -decay process is considered, then for the matrix element of this process we have

$$\begin{aligned}
& \langle f | S^{(2)} | i \rangle \\
&= \frac{(-i)^2}{2} \left( \frac{G_F}{\sqrt{2}} \right)^2 N_{p_1} N_{p_2} N_{k_1} N_{k_2} J_{\mu\nu}(p_1, p_2, k_1, k_2) \\
&\quad \times \bar{u}(p_1) \gamma_\mu (1 + \gamma_5) u(-k_1) \bar{u}(p_2) \gamma_\nu (1 + \gamma_5) u(-k_2) \\
&\quad - (p_1 \leftrightarrow p_2) - (k_1 \leftrightarrow k_2) + (p_1 \leftrightarrow p_2)(k_1 \leftrightarrow k_2),
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
J_{\mu\nu}(p_1, p_2, k_1, k_2) &= \int e^{-i(p_1+k_1)x_1} e^{-i(p_2+k_2)x_2} \\
&\quad {}_{out} \langle p_f | T(J_\mu(x_1) J_\nu(x_2)) | p_i \rangle_{in} dx_1 dx_2.
\end{aligned} \tag{2}$$

Here,  $N_p = (1/(2\pi)^{3/2})(1/(2p_0)^{1/2})$ ,  $p_1$  and  $p_2$  ( $k_1$  and  $k_2$ ) are four-momenta of electrons (antineutrinos),  $p_i$  and  $p_f$  are four-momenta of the initial and final nucleus.  $J_\mu(x)$  is the weak charged nuclear hadron current in the Heisenberg representation [24,25].

The matrix element in Eq. (1) contains the contributions from two subsequent nuclear beta decay processes and  $2\beta 2\nu$ -decay [26]. They could be separated, if we write the T-product of the two hadron currents as follows:

$$T(J_\mu(x_1)J_\nu(x_2)) = J_\mu(x_1)J_\nu(x_2) + \Theta(x_{20} - x_{10})[J_\nu(x_2), J_\mu(x_1)]. \quad (3)$$

The first term in the r.h.s. of Eq. (3) is associated with two subsequent nuclear beta decay processes, which are energetically forbidden for the most of  $2\beta 2\nu$ -decay isotopes. The second term corresponds to  $2\beta 2\nu$ -decay process. We see that the  $2\beta 2\nu$ -decay nuclear matrix element is given by the non-equal-time commutator of the two hadron currents. It will be shown later that this feature is crucial for our understanding of the approximation schemes of different nuclear models.

We further assume the following standard approximations: i) The non-relativistic impulse approximation for the hadronic current  $J_\nu(0, \vec{y})$ . ii) We keep only the contribution from the axial current. iii) Only the  $s_{1/2}$  wave states of the emitted electrons are considered. iv) Our interest will be restricted only to the most favored  $0_{initial}^+ \rightarrow 0_{final}^+$  nuclear transition. Then we have,

$$J_{\mu\nu}^{2\beta 2\nu}(p_1, p_2, k_1, k_2) = -i2M_{GT}\delta_{\mu k}\delta_{\nu k} \times 2\pi\delta(E_f - E_i + p_{10} + k_{10} + p_{20} + k_{20}), \quad k = 1, 2, 3, \quad (4)$$

where,

$$M_{GT} = \frac{i}{2} \int_0^\infty (e^{it(p_{10}+k_{10}-\Delta)} + e^{it(p_{20}+k_{20}-\Delta)}) M_{AA}(t) dt, \quad (5)$$

with

$$M_{AA}(t) = {}_f\langle 0^+ | \frac{1}{2} [A_k(t/2), A_k(-t/2)] | 0^+ \rangle_i, \quad A_k(t) = e^{iHt} A_k(0) e^{-iHt}. \quad (6)$$

Here,  $|0^+ \rangle_i$  and  $|0^+ \rangle_f$  are respectively the wave functions of the initial and final nuclei with their corresponding energies  $E_i$  and  $E_f$ .  $\Delta$  denotes the average energy  $\Delta = (E_i - E_f)/2$ .  $A_k(0)$  is the Gamow-Teller transition operator  $A_k(0) = \sum_i \tau_i^+ (\vec{\sigma}_i)_k$ ,  $k=1,2,3$ .

The time integral form of  $M_{GT}$  in Eqs. (5) and (6) has been the starting point for the methods, which avoid the explicit calculation of the intermediate nuclear states, e.g. the Operator Expansion Method (OEM) [15,16] [22,23] and the S-matrix approach [26]. The S-matrix approach requires the derivation of two-body operators from the corresponding exchange Feynmann diagrams and the calculation of the nuclear transition by using a given initial and final nuclear wave functions.

For an analytical study of the different approximation schemes, it is useful to rewrite the transition operator of the nuclear matrix element  $M_{AA}(t)$  in Eq. (5) into a infinite series of multiple commutators of the nuclear Hamiltonian  $H$  and the Gamow Teller transition operator  $A_k(0)$  with help of

$$A_k(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \overbrace{[H[H\dots[H, A_k(0)]\dots]]}^{n \text{ times}}. \quad (7)$$

If the multiple commutator is calculated without approximation for a nuclear Hamiltonian consisting of one- and two- body interactions, we obtain an infinite sum of many-body operators. This difficulty may be avoided if some approximation schemes are introduced, e.g. the QBA, the renormalized QBA or the approximation schemes of the Operator Expansion Method (OEM). We shall discuss this point in the next Section.

If we integrate over the time variable in Eq. (7)) using the standard procedure of the adiabatic switch-off of the interaction as  $t \rightarrow \infty$ , insert the complete set of the intermediate states  $|1_n^+ \rangle$  with eigenenergies  $E_n$  between the two axial currents in Eq. (6) and assume that the nuclear states are eigenstates of the nuclear Hamiltonian, we get the well-known form of  $M_{GT}$

$$M_{GT} = \frac{1}{2} \sum_n \langle 0_f^+ | A_k(0) | 1_n^+ \rangle \langle 1_n^+ | A_k(0) | 0_i^+ \rangle \times \left( \frac{1}{E_n - E_i + p_{10} + k_{10}} + \frac{1}{E_n - E_i + p_{20} + k_{20}} \right). \quad (8)$$

After the usual approximation  $p_{10} + k_{10} \approx p_{10} + k_{10} \approx \Delta$  the form of  $M_{GT}$  in Eq. (8) is suitable for the calculation within the commonly used intermediate nucleus approaches

(INA) to  $2\beta 2\nu$ -decay process like QRPA, RQRPA and shell model methods, which construct the spectrum of the intermediate nucleus by diagonalization.

### III. THE APPROXIMATION SCHEMES

#### A. Single particle Hamiltonian

Let restrict our consideration to a single particle Hamiltonian, which in the second quantization formalism takes the form

$$\hat{H}^{s.p.} = \sum_{p,m_p} e_p c_{p,m_p}^+ c_{p,m_p} + \sum_{n,m_n} e_n c_{n,m_n}^+ c_{n,m_n}. \quad (9)$$

Here,  $c_{p,m_p}^+$  and  $c_{n,m_n}^+$  ( $c_{p,m_p}$  and  $c_{n,m_n}$ ) are creation (annihilation) operators of proton and neutron, respectively and  $e_p$  and  $e_n$  are single particle energies of proton and neutron states. By using of the Eq. (7) and the anticommutation relation of particle operators for the time dependent axial current  $\hat{A}_k(t)$  we obtain

$$\hat{A}_k^{s.p.}(t) = \sum_{p,m_p,n,m_n} e^{i(e_p - e_n)t} \langle p, m_p | A_k(0) | n, m_n \rangle c_{p,m_p}^+ c_{n,m_n} \quad (10)$$

and

$$[\hat{A}_k^{s.p.}(t/2), \hat{A}_k^{s.p.}(-t/2)] = 0. \quad (11)$$

It means that if we consider only the single particle part of the nuclear Hamiltonian, the nuclear matrix  $M_{GT}$  is just equal to zero. It is however expected since the  $2\beta 2\nu$ -decay is a second order process correlated by the residual interaction. Without the residual interaction only the two-subsequent beta decay processes are possible, if they are energetically allowed. Clearly, without the residual interaction it is not possible to construct the spectrum of the intermediate nucleus.

We note that  $M_{GT}^{s.p.} = 0$  comes as a result of the cancelation between the direct and cross term of  $M_{GT}$  in Eq. (8). If we use  $E_n - E_i + p_{20} + k_{20} = E_n - E_f - p_{10} - k_{10}$  and transform  $M_{GT}$  in Eq. (8) in the integral representation we obtain



$$M_{GT} = \lim_{\epsilon \rightarrow 0} i \int_0^\infty \langle 0_f^+ | \hat{A}_k(0) \hat{A}_k(-t) e^{-it(p_{10}+k_{10}-\epsilon)} + \hat{A}_k(t) \hat{A}_k(0) e^{-it(-p_{10}-k_{10}-\epsilon)} | 0_i^+ \rangle dt. \quad (12)$$

If we suppose  $\hat{A}_k(t) \approx \hat{A}_k^{s.p.}(t)$ , integrate over time variable in Eq. (12) and use the anti-commutation relation of the particle operator we find a complete cancelation between both terms in the r.h.s. of Eq. (12). It shows that the single particle operator of the nuclear Hamiltonian plays a less important role in the evaluation of  $2\beta 2\nu$ -decay matrix elements. This situation has not been noticed in Ref. [27] in which the approximation  $\hat{H} \approx \hat{H}^{s.p.}$  was also discussed and therefore the authors there came to a different conclusion.

## B. The QRPA and RQRPA Hamiltonians

The INA approach for the calculation of the  $M_{GT}$  in Eq. (8) consists of two QRPA diagonalizations related to the initial and final nuclei. The corresponding initial and final QRPA Hamiltonians  $\hat{H}^i$  and  $\hat{H}^f$  take the forms

$$\begin{aligned} \hat{H}^i &= const_i + \sum_{m_i JM} \Omega^{m_i} Q_{JM}^{+m_i} Q_{JM}^{m_i}, \\ \hat{H}^f &= const_f + \sum_{m_f JM} \Omega^{m_f} Q_{JM}^{+m_f} Q_{JM}^{m_f}, \end{aligned} \quad (13)$$

which are connected with two sets of intermediate nuclear states

$$|m_i JM, i\rangle = Q_{JM}^{+m_i} |0_{qrpa}^+ \rangle_i \quad |m_f JM, f\rangle = Q_{JM}^{+m_f} |0_{qrpa}^+ \rangle_f \quad (14)$$

generated from initial and final nuclei, respectively. Henceforth we use label "i" for initial and "f" for the final nuclei.  $\Omega^{m_{i,f}}$  is the energy of the m-th intermediate state and the phonon creation operators  $Q_{JM}^{+m_{i,f}}$  is defined as followed:

$$\begin{aligned} Q_{JM}^{+m_i} &= \sum_{pn} (X_{(pn)J}^{m_i} A^+(pnJM) - Y_{(pn)J}^{m_i} \tilde{A}(pnJM)), \\ Q_{JM}^{+m_f} &= \sum_{pn} (X_{(pn)J}^{m_f} B^+(pnJM) - Y_{(pn)J}^{m_f} \tilde{B}(pnJM)). \end{aligned} \quad (15)$$

$X_{(pn)J}^{m_{i,f}}$  and  $Y_{(pn)J}^{m_{i,f}}$  are forwards- and backwards- variational amplitudes, respectively.  $A^+(pnJM)$  and  $B^+(pnJM)$  are respectively boson creation operators of initial and final nuclei.

The time dependent axial current  $A_k(t)$  can be expanded in the QRPA phonon basis as follows [27]:

$$\hat{A}_k^{QRPA}(t) = \sum_{pnm} \langle p || \sigma || n \rangle [(u_p v_n X_{(pn)1}^m + v_p u_n Y_{(pn)1}^m) e^{i\Omega_m t} Q_{1k}^{+m} + (v_p u_n X_{(pn)1}^m + u_p v_n Y_{(pn)1}^m) e^{-i\Omega_m t} \tilde{Q}_{1k}^m]. \quad (16)$$

Here,  $u$  and  $v$  are the BCS occupation amplitudes.

The expression in Eq. (16) allows us to calculate the transition operator of the time dependent nuclear matrix element  $M_{AA}(t)$  in Eq. (6). If we suppose the equivalence between both QRPA Hamiltonians in (13) we obtain

$$[\hat{A}_k^{QRPA}(t/2), \hat{A}_k^{QRPA}(-t/2)] = \text{const}, \quad (17)$$

which implies that the  $2\beta 2\nu$ -decay nuclear matrix element is equal to zero because the  $2\beta 2\nu$ -decay operator should be at least a two-body operator changing two neutrons into two protons. A generalization of the above discussion to the RQRPA approach is straightforward and leads to the same conclusion. It means that the suppression of the  $2\beta 2\nu$ -decay nuclear matrix element is connected with the fact that it is a higher order effect in the boson or renormalized boson expansion of the nuclear Hamiltonian. We can obtain non-zero results only if we go beyond the first order boson or renormalized boson Hamiltonians.

One can ask why non-zero results are obtained in the INA QRPA and RQRPA calculations of  $M_{GT}$ . We believe that the following reasons could be the origin of this problem: i) The initial and final QRPA ground states are not orthogonal. Therefore, even for a constant transition operator non-zero results could be obtained. ii) The particle number non-conservation. We note that even the average particle numbers of protons and neutrons for the excited states of the intermediate nucleus differ from the correct ones. iii) There is a mismatching between the initial and final QRPA Hamiltonians and as a consequence the two sets of intermediate nuclear states generated from initial and final nuclei are not orthogonal to each other. In the QRPA or RQRPA calculation of the  $M_{GT}$  one arrives at the formula:

$$M_{GT}^{2\nu} = \sum_{m_i, m_f} \frac{{}_f \langle 0_{qrp}^+ | \hat{A}_k(0) | 1_{m_f}^+ \rangle \langle 1_{m_f}^+ | 1_{m_i}^+ \rangle \langle 1_{m_i}^+ | \hat{A}_k(0) | 0_{qrp}^+ \rangle_i}{\Omega_{1+}^{m_f} + \Omega_{1+}^{m_i}} \quad (18)$$

Here,  $\langle 1_{m_f}^+ | 1_{m_i}^+ \rangle$  is the overlap factor of the intermediate nuclear states generated from initial and final nuclei, given by [2]

$$\langle 1_{m_f}^+ | 1_{m_i}^+ \rangle = \sum_{pn} (X_{(pn)1+}^{m_i} X_{(pn)1+}^{m_f} - Y_{(pn)1+}^{m_i} Y_{(pn)1+}^{m_f}). \quad (19)$$

This overlap factor has been considered practically in all QRPA or RQRPA calculations of  $2\beta 2\nu$ -decay process. In the case in which the two sets of intermediate nuclear states deduced from initial and final nuclei are identical Eq. (18) is just the orthonormal condition for two QRPA states. However, this is not the case in a realistic calculation and we shall show later that we can hardly expect it within the QBA. In addition we note that the phases of the two sets of intermediate states are in principal arbitrary. Therefore, it is necessary to identify them e.g. by requiring the diagonal elements of the overlap matrix to be positive or by requiring the largest component of the wave function for each state to be positive.

The equivalence of the two sets of the intermediate nuclear states is connected with the equivalence of both QRPA Hamiltonians in Eq. (13). Let discuss this point within the QBA.

The quasiparticle creation and annihilation operators of the initial ( $a^+, a$ ) and final ( $b^+, b$ ) nuclei are connected with the particle creation and annihilation ( $c^+, c$ ) operators by the BCS-transformations. As a consequence there is a unitary transformation between the initial and final quasiparticles both for protons and neutrons. In the case of proton quasiparticles it takes the form:

$$\begin{pmatrix} a_p^+ \\ a_{\bar{p}} \end{pmatrix} = \begin{pmatrix} u_p^i & -v_p^i \\ v_p^i & u_p^i \end{pmatrix} \begin{pmatrix} u_p^f & v_p^f \\ -v_p^f & u_p^f \end{pmatrix} \begin{pmatrix} b_p^+ \\ b_{\bar{p}} \end{pmatrix}, \quad (20)$$

$$\begin{pmatrix} \tilde{u}_p & -\tilde{v}_p \\ \tilde{v}_p & \tilde{u}_p \end{pmatrix} \begin{pmatrix} b_p^+ \\ b_{\bar{p}} \end{pmatrix},$$

Relation (20) allows us, by using the QBA, to rewrite the boson operators of the initial nucleus with the help of the boson operators of the final nucleus:

$$A^+(pnJM) = \tilde{u}_p \tilde{u}_n B^+(pnJM) - \tilde{v}_p \tilde{v}_n \tilde{B}(pnJM) \quad (21)$$

We note that  $|\tilde{u}| \approx 1$  and  $|\tilde{v}| \approx 0$ . Therefore, we shall omit further terms proportional to  $\tilde{v}\tilde{v}$ . Next, we can rewrite  $Q_{JM}^{+m_i}$  with the  $Q_{JM}^{+m_f}$  and  $\tilde{Q}_{JM}^{m_f}$  as follows:

$$Q_{JM}^{+m_i} = \sum_{m_f} (\mathcal{O}_J^{m_i m_f} Q_{JM}^{+m_f} + \mathcal{P}_J^{m_i m_f} \tilde{Q}_{JM}^{m_f}), \quad (22)$$

where,

$$\mathcal{O}_J^{m_i m_f} = \sum_{pn} \tilde{u}_p \tilde{u}_n (X_{(pn)J}^{m_i} X_{(pn)J}^{m_f} - Y_{(pn)J}^{m_i} Y_{(pn)J}^{m_f}), \quad (23)$$

$$\mathcal{P}_J^{m_i m_f} = \sum_{pn} \tilde{u}_p \tilde{u}_n (X_{(pn)J}^{m_i} Y_{(pn)J}^{m_f} - Y_{(pn)J}^{m_i} X_{(pn)J}^{m_f}). \quad (24)$$

It is now straightforward to rewrite  $H^i$  with the phonon operators of the final nucleus and to perform a comparison with the  $H^f$ . The equivalence of both Hamiltonians requires that the following relations are fulfilled:

$$\sum_{m_i} \Omega_J^{m_i} (\mathcal{O}_J^{m_i m_f} \mathcal{O}_J^{m_i m'_f} + \mathcal{P}_J^{m_i m_f} \mathcal{P}_J^{m_i m'_f}) = \Omega_J^{m_f} \delta_{m_f m'_f} \quad (25)$$

$$\sum_{m_i} \Omega_J^{m_i} (\mathcal{O}_J^{m_i m_f} \mathcal{P}_J^{m_i m'_f}) = 0. \quad (26)$$

We know that in the vicinity of the collapse of the QRPA solution the lowest  $1^+$  state of the intermediate nucleus plays an important role in the calculation of  $M_{GT}$  as it is strongly influenced by the ground state correlations. Therefore, there is an interest to check the validity of the expression (25) for this state. We note that the above expressions could be used also in the case of the renormalized QBA scheme, if we replace the  $X$  and  $Y$  amplitudes with the renormalized amplitudes  $\overline{X}$ ,  $\overline{Y}$  (see Ref. [10]) and suppose for the renormalized factors the following relation  $\mathcal{D}_{(pn)1^+}^i / \mathcal{D}_{(pn)1^+}^f \simeq 1$ . The lowest energy of the intermediate state  $\Omega_{1^+}^{m_f}$  for  $2\beta 2\nu$ -decay of  $^{76}\text{Ge}$  obtained directly from the final nucleus by the QRPA and RQRPA diagonalization and indirectly with help of expression (25) for three different model spaces given in Ref. [28] is presented in Fig. 1. By glancing at Fig. 1. we see that close to the collapse of the pn-QRPA the equivalence of the initial and final QRPA Hamiltonians is getting worse with increasing parameter  $g_{pp}$ . It is more apparent if a larger model space

is used. We remark that the accuracy of the pn-RQRPA calculation of the lowest  $1^+$  state for a physically acceptable value of  $g_{pp}$  is considerably better in respect to the QRPA one. It clearly shows that the RQRPA offers a more reliable solution. However, the initial and final RQRPA Hamiltonians still remain different and this could be one of the reasons of the non-zero results obtained in the INA calculation of  $M_{GT}$ .

For the sake of completeness we also plot in Fig. 2 the  $M_{GT}$  of the  $2\beta 2\nu$ -decay of  $^{76}\text{Ge}$  calculated within the INA pn-QRPA and pn-RQRPA approach for three different model spaces. We see that in both cases  $M_{GT}$  demonstrates an increased sensitivity to  $g_{pp}$  with enhanced model space. We note also that the pn-RQRPA allows to perform calculations far behind the collapse of the pn-QRPA. These calculations have been performed with an overlap factor of the initial and final states, which has been approximated as follows:

$$\langle 1_{m_f}^+ | 1_{m_i}^+ \rangle \approx [Q_{JM}^{m_f}, Q_{JM}^{m_i}] = \mathcal{O}_J^{m_f m_i} \quad (27)$$

The advantage of this overlap factor over the one of Eq. (19) is that now the results are independent of the phases of the quasiparticle states, which are in principal arbitrary [For a given quasiparticle eigenenergy  $E$ , there are two solutions  $(u, v)$  and  $(-u, -v)$ ]. For the overlap factor of Eq. (19) this condition is not fulfilled. This can be easily proved by a numerical test.

It is also of interest to evaluate  $M_{GT}$  by using the overlap functions  $\mathcal{O}_1^{m_i m_f}$  and  $\mathcal{P}_1^{m_i m_f}$  and only one of the two RQRPA Hamiltonian, e.g. the  $\hat{H}^i$ . With the help of Eqs. (16) and (22) one has for the  $\beta^+$  transition amplitudes:

$$\langle 1_{m_i}^+ | \hat{A}(0) | 0_{grpa}^+ \rangle_i = \frac{1}{\sqrt{3}} \sum_{pn} \sqrt{D_{(pn)1}^i} [u_p^i v_n^i \bar{X}_{(pn)1}^{m_i} + v_p^i u_n^i \bar{Y}_{(pn)1}^{m_i}] \quad (28)$$

$$\begin{aligned} {}_f \langle 0_{grpa}^+ | \widetilde{\hat{A}(0)} | 1_{m_f}^+ \rangle = & \frac{1}{\sqrt{3}} \sum_{pn m_i} [(v_p^i u_n^i \bar{X}_{(pn)1}^{m_i} + u_p^i v_n^i \bar{Y}_{(pn)1}^{m_i}) \mathcal{O}_1^{m_i m_f} \\ & + (u_p^i v_n^i \bar{X}_{(pn)1}^{m_i} + v_p^i u_n^i \bar{Y}_{(pn)1}^{m_i}) \mathcal{P}_1^{m_i m_f}] \sqrt{D_{(pn)1}^i}. \end{aligned} \quad (29)$$

The  $M_{GT}$  calculated with help of only one pn-RQRPA Hamiltonian is drawn in Fig. 3. We note a large discrepancy between the results obtained with initial and final pn-RQRPA

Hamiltonians and those obtained by the standard INA calculation. In this way one sees that the  $2\beta 2\nu$ -decay matrix elements are very sensitive to the restrictions of the RQBA approximation. From the Fig. 3 it follows also that the INA calculation has been mostly influenced by the final pn-RQRPA Hamiltonian. It is an indication that the main problem of the INA consist in the non-orthogonality of the initial and final ground states.

The main message of this Section is that the  $2\beta 2\nu$ -decay is a higher order effect in the boson expansion of the nuclear Hamiltonian. Several procedures have been proposed which outline the importance of higher order effects and try to take into account some higher-order QRPA corrections [5]. Maybe, the most perspective way is the fermion-boson mapping procedure discussed by M. Sambataro, F. Catara and J. Suhonen [29] within a schematic model. In this way it is expected that the correspondence between the initial and final nuclear Hamiltonians will be improved and more reliable results could be obtained.

It is worthwhile to notice that the usual strategy has been first to try to reproduce the observed  $2\beta 2\nu$  - decay half times within a given nuclear model in order to gain confidence in the calculated  $2\beta 0\nu$  - decay nuclear matrix elements. However, there is a principal difference from the nuclear physics point of view. The  $0\beta 2\nu$  - decay is not a higher order effect in the boson expansion of the nuclear Hamiltonian and therefore the renormalized QBA scheme could be sufficient. It is because the nucleons undergoing beta decays in the nucleus are correlated by the exchange of Majorana neutrinos and then the decomposition in Eq. (3) is irrelevant. We remind the reader that the QRPA and RQRPA Hamiltonians have been found successful in describing the single beta decay transition.

### C. The Operator Expansion Method

The OEM is a nuclear structure method for the  $2\beta 2\nu$ -decay, which has the advantage to avoid the explicit sum over the intermediate nuclear states. There are two different ways to derive the  $2\beta 2\nu$ -decay OEM transition operators. In an approach proposed by Ching and Ho (OEM1) [14] the expansion of the denominators of  $M_{GT}$  in Eq. (12) in Taylor series is

used. In a different approach proposed by Šimkovic (OEM2) [15] the OEM is derived from the integral representation of the nuclear matrix element  $M_{GT}$  in Eqs. (5), (6) and (7). It has been found that the OEM2 offers advantages over OEM1 as there are no problems of convergence in the power series expansion of the denominator, which has been a subject of criticism, J. Engel et al. [30].

In the OEM the  $2\beta 2\nu$ -decay transition operator is described by an infinite series of multiple commutators of the nuclear Hamiltonian  $H$  and the Gamow-teller transition operator  $A(0)$  [see Eq. (6) and (7)]. The OEM is based on two main assumptions [14,15]: i) It is assumed that the kinetic energy operator can be ignored in the resulting commutators and therefore the nuclear Hamiltonian is represented only by the two-body interaction terms. ii) Only two-body terms are retained in evaluating each commutator and higher order terms are neglected. These two approximations seem reasonable. The omission of the one-body terms of the nuclear Hamiltonian is justified by the fact that these terms play a less important role in the evaluation of  $M_{GT}$ . It is easy to see it if we consider only the one-body part of the nuclear Hamiltonian  $H_0$ , then  $A(t) = e^{iH_0 t} A(0) e^{-iH_0 t}$  as well as the commutator  $[A(t/2), A(-t/2)]$  are one body operators. However, there are no contributions to  $2\nu 2\beta$ -decay from a one body operator, as the  $2\nu 2\beta$ -decay operator should be at least a two-body operator changing two neutrons into two protons. In respect to the second approximation from the discussion in Section 3 it follows that this approximation goes beyond the QBA or renormalized QBA. In the case of the QBA the commutator  $[A(t/2), A(-t/2)]$  is just a constant but within the OEM the commutator  $[A(t/2), A(-t/2)]$  is a two-body transition operator changing two neutrons into two protons. So, it is not true that the OEM is an approximation to the QRPA as it was believed before [20].

Recently, the OEM has been reconsidered [23]. It has been shown that the Coulomb interaction plays a decisive role within the OEM. Gmitro and Šimkovic [16] were first to introduce a Coulomb interaction term

$$V_C = \frac{1}{2} \sum_{i \neq j} g_c(r_{ij}) O_{ij}^\tau, \quad O_{ij}^\tau = \frac{1}{4} (1 + \tau_i^0)(1 + \tau_j^0), \quad (30)$$

in the OEM formalism. Later it was done by Muto [20], who however obtained different OEM transition operators. We shall discuss this important point and explain the origin of these differences.

The time dependent nuclear matrix element  $M_{AA}(t)$  in (7) can be transformed to the following form:

$$M_{AA}(t) = e^{i(E_i - E_f)\alpha} \langle 0_f^+ | e^{iH\alpha} \frac{1}{2} [A_k(t/2), A_k(-t/2)] e^{-iH\alpha} | 0_i^+ \rangle. \quad (31)$$

Obviously  $M_{AA}(t)$  does not depend on  $\alpha$ , if the nuclear states and their corresponding energies can be considered as the eigenstates and eigenvalues of the nuclear Hamiltonian  $H$ .

$$H|0_i^+ \rangle = E_i|0_i^+ \rangle, \quad H|0_f \rangle = E_f|0_f \rangle \quad (32)$$

Using the machinery of the OEM for the operator inside the brackets in the r.h.s. in Eq. (31) one obtains

$$M_{AA}(t) = e^{i(E_i - E_f)\alpha} \langle 0_f^+ | \mathcal{P} \sum_{n \neq m} e^{ig_c(r_{nm})O_{nm}^\tau \alpha} \times \\ \tau_n^+ \tau_m^+ \mathcal{V}^{OEM}(t, r_{nm}, \Pi_s^\sigma(n, m), \Pi_t^\sigma(n, m), S_{nm}) | 0_i^+ \rangle. \quad (33)$$

Here,  $\Pi_s^\sigma$  and  $\Pi_t^\sigma$  are projectors onto spin singlet and triplet states and  $\mathcal{V}^{OEM}$  is the two-body OEM potential.  $\mathcal{P}$  denotes the Principle value integration. It is apparent that  $M_{AA}(t)$  is  $\alpha$  dependent for  $\alpha \neq 0$ . It means that if the nuclear Hamiltonian does not contain a Coulomb term [14,15], [17]–[19] or contains a Coulomb interaction in the form of Eq. (8) [16,20], [21], the derivation of the OEM transition operator is inconsistent. For  $\alpha = t/2$  we obtain the OEM transition operators of Muto [20] and the formulae of Gmitro and Šimkovic [16] could be reproduced for  $\alpha = 0$ , which differ from each other.

It is worthwhile to notice that the above results point out some more important aspects. The analysis is clearly showing that the mass difference of the initial and final nuclei  $E_i - E_f$  plays an important role in the calculation of  $2\nu 2\beta$ -decay.

The above mentioned inconsistency of the OEM could be avoided if one considers an effective Coulomb interaction term  $V_C$  [22]



$$V_C = \frac{1}{2} \sum_{i \neq j} (E_f - E_i) O_{ij}^\tau. \quad (34)$$

In this way the one-body terms of the nuclear Hamiltonian are not totally neglected.

Let's consider the approximated two-body nuclear Hamiltonian  $H$  containing central  $V_{CN}$  and tensor  $V_{TN}$  interactions in addition to the effective Coulomb interaction (the notation of Ref. [20] is used):

$$H \approx V_C + V_{CN} + V_{TN}, \quad (35)$$

where

$$V_{CN} = \frac{1}{2} \sum_{i \neq j} [ (g_{SE}(r_{ij}) \Pi_e^r(ij) + g_{SO}(r_{ij}) \Pi_o^r(ij)) \Pi_s^\sigma(ij) + (g_{TE}(r_{ij}) \Pi_e^r(ij) + g_{TO}(r_{ij}) \Pi_o^r(ij)) \Pi_t^\sigma(ij) ], \quad (36)$$

$$V_{TN} = \frac{1}{2} \sum_{i \neq j} (g_{TNE}(r_{ij}) \Pi_e^r(ij) + g_{TNO}(r_{ij}) \Pi_o^r(ij)) S_{ij}. \quad (37)$$

Then, within the OEM approximations the infinite series of the commutators in Eq. (7) could be summed using the formulae [15,16]

$$e^{igPt} A e^{-igPt} = \frac{1}{2} (A + PAP + \cos(2gt)(A - PAP) + i \sin(2gt)[P, A]), \quad (38)$$

$$e^{igOt} A e^{-igOt} = A - [O, [O, A]] + \cos(gt)[O, [O, A]] + i \sin(gt)[O, A], \quad (39)$$

for  $P^2 = 1$  and  $O^2 = O$ . Then by performing the integration in Eq. (5) over  $t$  we obtain the nuclear matrix element  $M_{GT}$ :

$$M_{GT} = \langle 0_f^+ | \frac{1}{2} \mathcal{P} \sum_{i \neq j} \tau_i^+ \tau_j^+ ( \mathcal{V}^{singlet}(r_{ij}) \Pi_s^\sigma(ij) + \mathcal{V}^{triplet}(r_{ij}) \Pi_t^\sigma(ij) + \mathcal{V}^{tensor}(r_{ij}) S_{ij} ) | 0_i^+ \rangle, \quad (40)$$

where,

$$\mathcal{V}^{singlet} = \frac{-2}{g_{TE} - g_{SE} - 4g_{TNE} + \Delta} - \frac{4}{g_{TE} - g_{SE} + 2g_{TNE} + \Delta}$$

$$\mathcal{V}^{triplet} = \frac{1}{3} \left[ \frac{4}{\Delta} + \frac{4}{-6g_{TNO} + \Delta} + \frac{4}{6g_{TNO} + \Delta} - \right]$$

$$\mathcal{V}^{tensor} = \frac{2}{g_{SO} - g_{TO} + 4g_{TNO} + \Delta} - \frac{4}{g_{SO} - g_{TO} - 2g_{TNO} + \Delta},$$

$$\frac{1}{3} \left[ \frac{1}{\Delta} + \frac{1}{-6g_{TNO} + \Delta} - \frac{2}{6g_{TNO} + \Delta} + \frac{1}{g_{SO} - g_{TO} + 4g_{TNO} + \Delta} - \frac{1}{g_{SO} - g_{TO} - 2g_{TNO} + \Delta} \right]. \quad (41)$$

If we neglect the tensor potential  $V_{TN}$  we have

$$M_{GT} = \langle 0_f^+ | \frac{1}{2} \mathcal{P} \sum_{n \neq m} \tau_n^+ \tau_m^+ (f^0(r) \Omega_{nm}^0 + f^1(r) \Omega_{nm}^1) | 0_i^+ \rangle \quad (42)$$

$$f^0(r) = \frac{-6}{g_{TE}(r) - g_{SE}(r) + \Delta}$$

$$f^1(r) = \frac{4}{\Delta} - \frac{2}{g_{SO}(r) - g_{TO}(r) + \Delta} \quad (43)$$

If we neglect both central and tensor potential  $V_{TN}$  we obtain

$$M_{GT} = \langle 0_f^+ | \frac{1}{2} \sum_{n \neq m} \tau_n^+ \tau_m^+ \frac{2}{\Delta} \vec{\sigma}_n \cdot \vec{\sigma}_m | 0_i^+ \rangle. \quad (44)$$

We note that the OEM transition operators obtained by Muto [20] differ from the above ones. This is due to the inconsistent derivation in the framework of the OEM1. The approximate Hamiltonian considered by Muto does not allow to fix the difference of the masses of the initial and final nuclei, which is a basic feature of the  $2\beta 2\nu$ -decay matrix element. Thus, one obtains different transition operators for different values of the parameter  $\alpha$  in Eq. (33). This important result is coming from the Heisenberg nature of the axial current operators and it can be hardly seen within the OEM1 method proposed by Ching and Ho [14]. The importance of the mass difference between the initial and final nuclei becomes apparent within the OEM2 [22], which contains elements of the field theory. In addition, the OEM2 shows explicitly that only the Principal value part of the  $\mathcal{V}^{OEM}$  potential is relevant for the nuclear matrix element  $M_{GT}$ . A fact which is not clear within the OEM1. We note that the poles of the  $\mathcal{V}^{OEM}$  potential appear for  $r_{nm} \approx 1.5 - 2.0$  fm. It is because of the small energy release for this process ( $\Delta \approx 2$  MeV). It means that only the long range part of the nucleon-nucleon interaction plays an important role in the calculation and that the

$\mathcal{V}^{OEM}$  potential is expected to be independent of the chosen type of the nucleon-nucleon interaction. It is worthwhile to notice that the consistent OEM-potential in Eqs. (40)-(44) does not vanish even for a zero-range  $\delta$ -force.

For the calculation of  $M_{GT}$  in Eq. (39) it is necessary to know the wave functions of the initial and final nucleus. The OEM could be combined with the ground state wave functions of the QRPA or renormalized RQRPA models [17]– [21] (OEM+RPA)

$$M_{GT}^{OEM+RPA} = \sum_{\substack{pn\bar{p}\bar{n} \\ J^\pi m_i m_f \mathcal{J}}} (-)^{j_i+j_{k'}+J+\mathcal{J}} (2\mathcal{J}+1) \begin{Bmatrix} j_p & j_n & J \\ j_{n'} & j_{p'} & \mathcal{J} \end{Bmatrix} \times \\ \langle f | 0_{rpa}^+ \parallel [c_p^+ \widetilde{c}_{n'}]_J \parallel J^\pi m_f \rangle \langle J^\pi m_f | J^\pi m_i \rangle \langle J^\pi m_i \parallel [c_p^+ \tilde{c}_n]_J \parallel 0_{rpa}^+ \rangle_i \\ \times \langle p, p'; \mathcal{J} | \mathcal{P} \tau_1^+ \tau_2^+ \mathcal{V}^{OEM}(t, r_{12}, \Pi_s^\sigma(1, 2), \Pi_t^\sigma(1, 2), S_{1,2}) | n, n'; \mathcal{J} \rangle \quad (45)$$

In the previous OEM+QRPA calculations with the inconsistent OEM potential  $\mathcal{V}^{OEM}$ , the  $M_{GT}^{OEM+RPA}$  has been found not sensitive to the strength of the particle-particle interaction [17]– [20]. Recently, the  $M_{GT}^{OEM+RPA}$  has been calculated for the  $2\beta 2\nu$ -decay of  $^{76}\text{Ge}$  within the renormalized pn-QRPA with the overlap factor in Eq. (19) and a consistent OEM-potential in Eqs. (41)-(41) [23]. Small and large model spaces containing respectively the full  $2 - 4\hbar\omega$  the full  $0 - 5\hbar\omega$  major oscillator shells have been considered. A strong suppression of the results with increasing  $g_{pp}$  has been found [23]. The sensitivity of  $M_{GT}^{OEM+RPA}$  to  $g_{pp}$  within the physically acceptable region of particle-particle strength has been increased considerably with the enhancement of the model space. It is clear that this effect could have its origin only in the pn-RQRPA wave functions. We recall that the OEM-potential is a two-body operator, which represents a sum over all intermediate nuclear states and it is independent of the basis. Then, the instability of the results has to be related to the overlap factor and to the fact that the initial and final states are not orthogonal.

We mentioned already that the overlap factor of Eq. (19) does not guarantee the independence of the results in respect to the phases of the BCS states, which are arbitrary. We have found that the  $2\beta 2\nu$ -decay matrix element is very sensitive to this problem. One can see it by calculating  $M_{GT}^{OEM+RPA}$  with different overlap factors, i.e. a delta function overlap

factor and the overlap factors of Eqs. (19) and (27). The uncertainty of the results is build in into the calculation through the pn-RQRPA wave functions. It is worth to notice that a similar problem appears by calculating the closure matrix element,

$$M_{clos} = \langle 0_f^+ | \sum_{n \neq m} \tau_n^+ \tau_m^+ \vec{\sigma}_n \cdot \vec{\sigma}_m | 0_i^+ \rangle . \quad (46)$$

We note that  $M_{clos}$  differs only by a constant with the  $M_{OEM}$  in the case in which central and tensor interactions are neglected. From the above discussion it follows that OEM should be not combined with QRPA or RQRPA. Nevertheless, there is an interest for OEM calculation with other ground state wave functions especially with shell model ones.

#### IV. CONCLUSIONS

In summary, the integral representation of the nuclear matrix element  $M_{GT}$  in Eq. (5), (6) and (7) has been found useful for a critical analysis of different approximation schemes. The single-particle approximation of the nuclear Hamiltonian implies  $M_{GT}$  to be equal to zero because of the mutual cancelation of the direct and cross terms. It means that without the residual interaction there is no  $2\beta 2\nu$ -decay and only the two subsequent beta decays are possible, if they are energetically allowed. We note that without the residual interaction no excited states could be generated.

The use of the pn-QRPA and pn-RQRPA nuclear Hamiltonians reveals that the  $2\beta 2\nu$ -decay transition operator is a constant, if one assumes that the initial and final nuclear Hamiltonians correspond to each other. However,  $2\beta 2\nu$ -decay operator should be at least a two-body operator changing two neutrons into two-protons. Therefore, the inclusion of higher order terms of the boson expansion of the nuclear Hamiltonian is necessary for a QRPA and RQRPA treatment of the  $2\beta 2\nu$ -decay process. It is worth mentioning that there is no such requirement in the case of the single beta decay and  $2\beta 0\nu$ -decay calculations.

By using a unitary transformation between the quasiparticles of the initial and final uncorrelated BCS ground states the problem of the equivalence of the initial and final pn-QRPA (pn-RQRPA) nuclear Hamiltonians has been studied. It has been found that the

pn-RQRPA Hamiltonians demonstrate a better mutual agreement like the pn-QRPA ones. The mismatching between both pn-QRPA Hamiltonians is large and their correspondence is getting worse with increasing model space and particle-particle strength parameter  $g_{pp}$ .

Previously, it was believed that the OEM is an approximation to the QRPA approach [20]. However, the  $2\beta 2\nu$ -decay transition operator within the OEM is a two-body operator and not a constant as within the QRPA or RQRPA. It means that the OEM approximations go beyond the QBA and RQBA. We note that the previous derivation [20] of the  $2\beta 2\nu$ -decay OEM transition operator was inconsistent because the role of the energy difference between the initial and final nuclear ground state in the calculation was overlooked. We present the OEM potential derived in a consistent way by considering the effective Coulomb interaction term. In this way the one-body terms of the nuclear Hamiltonian are not totally neglected. A combination of the OEM transition operator with the pn-RQRPA wave functions reflect the instabilities incorporated in the pn-RQRPA treatment of the two-vacua problem. Therefore, we suppose that the combination of the OEM-transition operator with other ground state wave functions, e.g. shell model ones, could be perhaps more predictive for the  $2\beta 2\nu$ -decay process.

We remark that the OEM is a special method developed for the  $2\beta 2\nu$ -decay, which could perhaps find a wider use for the study of the pion p-wave contribution to the process of Double Charge Exchange (DCX) pion on nuclei. The application of the OEM method is limited to the problems, where the nucleon-nucleon correlations by a meson exchange plays a dominant role. However, this is not the case in single beta decay and neutrinoless double beta decay calculations (there is a neutrino correlation of two beta decays in the nucleus). For these processes the one-body term of the nuclear Hamiltonian is expected to play a crucial role. Therefore, methods constructing explicitly the intermediate nucleus spectrum could be more successful in the treatment of these processes.

The presented studies have shown that the two-vacua problem appearing in the calculations of the  $2\beta 2\nu$ -decay matrix elements are not safely treated within the QBA or RQBA. As a result there are two-sets of intermediate nuclear states generated from initial and final

nuclei, which do not correspond to each other. It is because of the particle number non-conservation and the violation of the Pauli exclusion principle. Both these problems are connected to each other because one can not guarantee the particle number conservation without an explicit consideration of the Pauli principle. We note that the RQBA takes into account the Pauli principle only in an approximate way. In the QRPA and RQRPA calculation of the  $2\beta 0\nu$ -decay matrix element the inaccuracy of the considered approximations for solving the two-vacua problem is covered by the introduced overlap factor. In this paper we have proposed a less critical way to evaluate it. We believe that by including higher order terms of the boson expansion of the nuclear Hamiltonian the equivalence of both nuclear Hamiltonians and of the two sets of the intermediate nuclear states will be improved and the overlap factor will be closer to a delta function. A schematic study of M. Sambataro et al [29], based on the fermion-boson mapping procedure is rather encouraging and could lead to a better agreement of the two-different nuclear Hamiltonians and to more reliable results.

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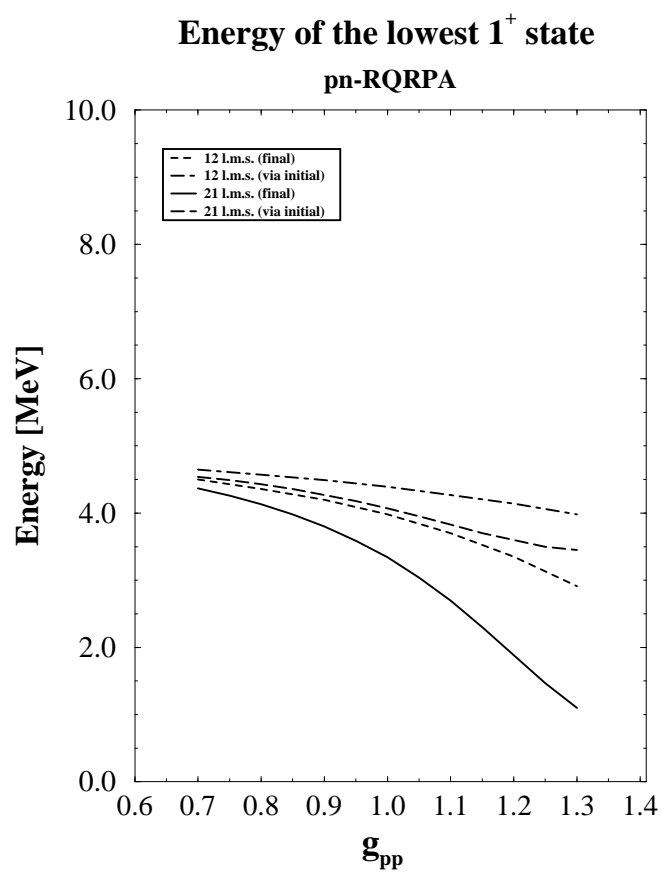
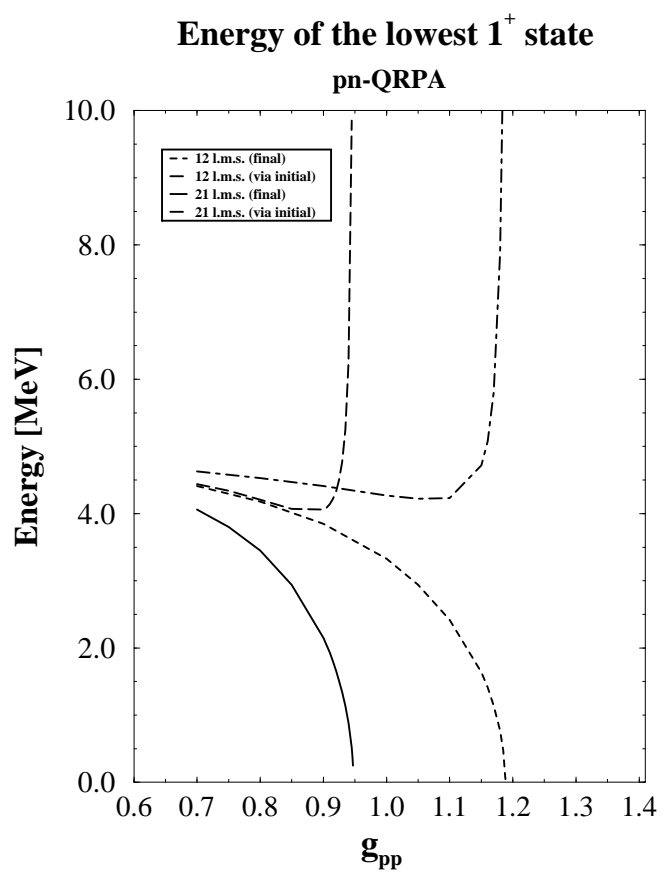
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## FIGURES

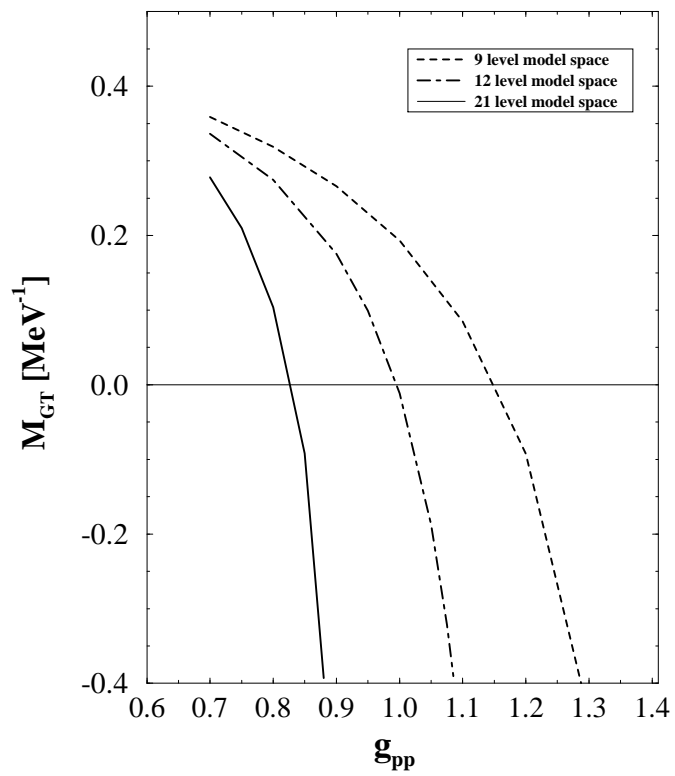
FIG. 1. The energy of the lowest  $1^+$  state of the intermediate nucleus  $^{76}\text{As}$  calculated directly within the pn-QRPA and pn-RQRPA for  $^{76}\text{Se}$  isotope and indirectly with the help of Eq. (25) is plotted as function of the particle-particle coupling constant  $g_{pp}$  for a 12-level (the full  $2 - 4\hbar\omega$  major oscillator shells) and a 21-level model space (the full  $0 - 5\hbar\omega$  major oscillator shells).

FIG. 2. The nuclear matrix element  $M_{GT}$  for the  $2\nu 2\beta$ -decay of  $^{76}\text{Ge}$  calculated within the pn-QRPA and pn-RQRPA INA, is plotted as a function of the particle-particle coupling constant  $g_{pp}$ . The dashed line corresponds to the 9-level model space (the full  $3 - 4\hbar\omega$  major oscillator shells), the dot-dashed line to the 12-level model space (the full  $2 - 4\hbar\omega$  major oscillator shells) and the solid line to the 21-level model space (the full  $0 - 5\hbar\omega$  major oscillator shells).

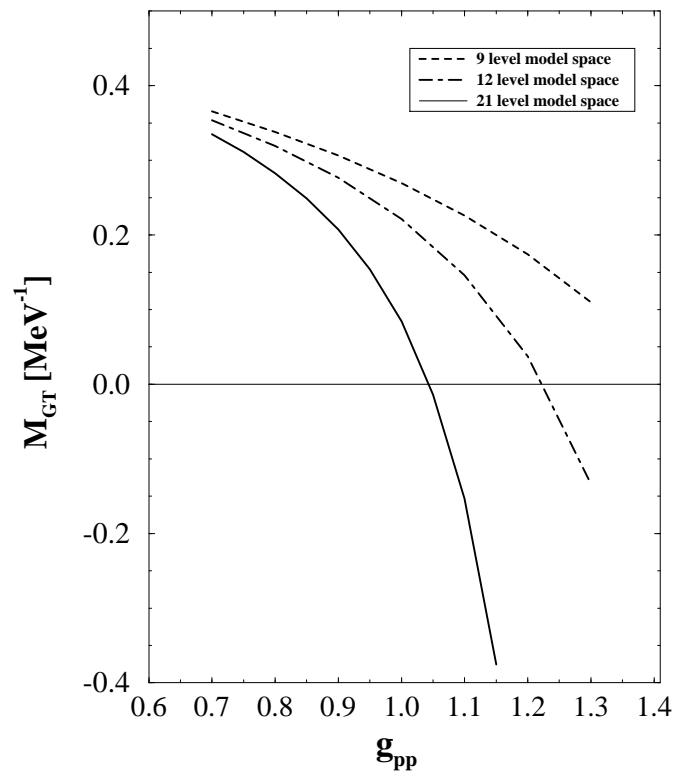
FIG. 3. The calculated nuclear matrix element  $M_{GT}$  for the  $2\nu 2\beta$ -decay of  $^{76}\text{Ge}$  pn-RQRPA INA plotted as a function of the particle-particle coupling constant  $g_{pp}$  for a 12-level and a 21-level model space. The solid line corresponds to the standard calculation with the initial and final QRPA Hamiltonians. The dashed line is the calculation which considers only the initial QRPA Hamiltonian and the overlap functions [see Eqs. (28) and (29)] and the dot-dashed line the final QRPA Hamiltonian and the overlap functions.



pn-QRPA:  $^{76}\text{Ge}$

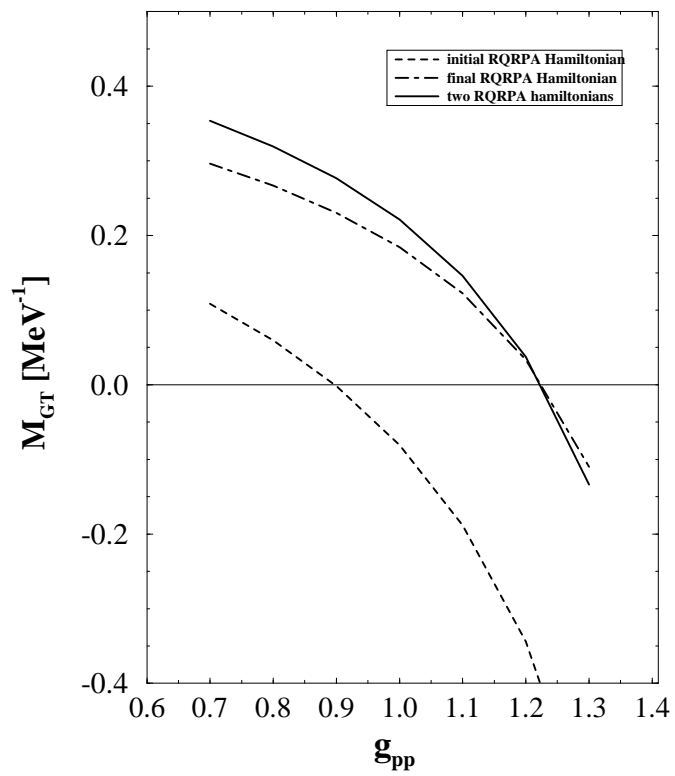


pn-RQRPA:  $^{76}\text{Ge}$



**pn-RQRPA:  $^{76}\text{Ge}$**

12-level model space



**pn-RQRPA:  $^{76}\text{Ge}$**

21-level model space

